



# **Intrinsic Rates and Stable Age-Specific Mortality (and Migration) Rates of Growth Matrix Operator in Single Region (Multiregion) Population Model**

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INTRINSIC RATES AND STABLE AGE-SPECIFIC MORTALITY  
(AND MIGRATION) RATES OF THE GROWTH MATRIX OPERATOR  
IN THE SINGLE REGION (MULTIREGION) POPULATION MODEL

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## Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

- I. the study of spatial population dynamics;
- II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;
- III. the analysis and design of migration and settlement policy;
- IV. a comparative study of national migration and settlement patterns and policies.

This paper, the twelfth in the spatial population dynamics series, focuses on the schedule of mortality (migration) embodied in the stable state of single-region (multiregion) models of population projection. It points out that age-specific mortality (migration) rates take on stable values that are slightly different from their initial values. Appropriate formulas for these stable rates are derived here and then used for a direct derivation of intrinsic death (outmigration) rates.

Related papers and other publications of the migration and settlement study are given on the back page of this report.

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## Abstract

This paper attempts to characterize the schedule of demographic events embodied in the stable state of single-region and multiregion models of population projection. The focus is on the age-specific mortality (migration) rates that reach, at stability, limiting values slightly different from their initial ones. Formulas giving the stable age-specific death (and migration) rates are provided as well as an indication of the discrepancy between initial and stable rates. Also, a direct calculation method for the computation of intrinsic rates relating to each demographic event is developed. Finally, in the multi-region case, the stable migration rates are compared to those derived earlier by Stone.

## Acknowledgements

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Intrinsic Rates and Stable Age-Specific Mortality  
(and Migration) Rates of the Growth Matrix Operator  
in the Single-Region (Multiregion) Population Model

Introduction

It is well-known that single-region and multiregion models of population projection eventually lead to a stable state. For example, consider the following quotations from Keyfitz and Rogers:

"If a population maintains fixed rates of fertility and survivorship at all ages, it will gradually approach a condition in which its age composition...remains constant" (Keyfitz 1965) (underline added).

"The population system..., if subjected to the unchanging schedule of fertility, mortality and mobility that is defined by the growth matrix in that equation, ultimately will increase at a constant stable growth rate." (Rogers 1966).\*

Such statements cannot be considered true anymore if fixed age-specific mortality (and migration) rates are substituted for fixed rates of survivorship or schedules of mortality (or migration), because the age-specific mortality (and migration) rates vary over the projection period as a consequence of the change in the age composition of the population. When stability is reached, these rates take on limiting values that are slightly different from the initial ones. These limiting values may be derived using the continuous formulation of the demographic projection models.

In this paper, the formulas for stable age-specific mortality (and migration) rates are presented and an indication of their discrepancy with the initial values is given. Part II continues with the direct computation of the intrinsic rates in both the single-region and the multiregion case. Finally, in the multiregion case, the formulas for migration rates are compared with those developed by Stone (1968).

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\*The general phrase "unchanging schedule" is all the more ambiguous because there exists a clear divergence between the assumptions relating to the fertility pattern, on the one hand, and to the mortality (migration) pattern on the other. The fertility pattern arises from an assumption of constant average age-specific fertility rates (implying changing instantaneous rates) whereas the death (migration) pattern is characterized by constant survivorship proportions (implying unchanging instantaneous rates).

### Age-Specific Mortality (and Migration) Rates at Stability

The derivation of stable age-specific mortality rates in the single region case requires consideration of the continuous formulation of the projection model. One of the basic assumptions of this model is that the schedule of mortality in terms of the instantaneous death rates, or forces of mortality,  $\mu(y)$ , remains unchanged. However, this hypothesis is not equivalent to one of constant age-specific death rates. This may be seen from the continuous version of the formula defining age-specific death rates  $\hat{M}_x$  between ages  $x$  and  $x + T$ , where  $T$  is the length of the age groups, and the duration of each projection interval:

$$\hat{M}_x^T = \frac{\int_0^T \mu(x+t) k(x+t) dt}{\int_0^T k(x+t) dt} \quad (1)$$

where  $k(y)$  is the population aged  $y$ .

At stability,

$$k(y) = e^{-ry} \ell(y),$$

where  $r$  is the stable growth rate and  $\ell(y)$  is the number of survivors of the initial radix  $\ell(0)$  in the life table constructed using the mortality schedule defined by the curve  $\mu(y)$ .

Then,

$$\hat{M}_x^T = \frac{\int_0^T e^{-r(x+t)} \mu(x+t) \ell(x+t) dt}{\int_0^T e^{-r(x+t)} \ell(x+t) dt} \quad (2)$$

Since  $\mu(y)$  is equal to  $-\frac{d\ell(y)}{\ell(y)dt}$  (Keyfitz 1968, p. 5), the numerator of (2) may be rewritten as:

$$\int_0^T e^{-r(x+t)} d[\ell(x+t)].$$

Integrating by parts,

$$\hat{T}^M_x = \frac{e^{-rx} \ell(x) - e^{-r(x+T)} \ell(x+T) - r \int_0^T e^{-r(x+t)} \mu(x+t) \ell(x+t) dt}{\int_0^T e^{-r(x+t)} \ell(x+t) dt}$$

or:

$$\hat{T}^M_x = \frac{e^{-rx} \ell(x) - e^{-r(x+T)} \ell(x+T)}{\int_0^T e^{-r(x+t)} \ell(x+t) dt} - r \quad (3)$$

How does  $\hat{T}^M_x$  compare with the initial death rate  $T^M_x$ ? The answer to this question is obtained by expressing the life table quantities in equation (3) as functions of  $T^M_x$ .

The initial death rates  $T^M_x$  are also those of the life table based on the mortality schedule  $\mu(y)$ :

$$T^M_x = \frac{\ell(x) - \ell(x+T)}{L(x)}, \quad (4)$$

$$\text{where } L(x) = \int_0^T \ell(x+t) dt$$

(Note that (4) is just (3) with  $r$  set equal to zero).

Comparison of  $T^M_x$  and  $\hat{T}^M_x$  requires an integration method for the denominators of (3) and (4).

Generally, a linear integration is used to derive  $L(x)$ :

$$L(x) = \frac{T}{2} [\ell(x) + \ell(x+T)], \quad (5)$$

and it is customary to assume that

$$\int_0^T e^{-r(x+t)} \ell(x+t) dt = e^{-r(x+\frac{T}{2})} L(x) * \quad (6)$$

Combining these last two formulas with (3) and (4) gives the formula relating  $\hat{T}^M_x$  to  $T^M_x$ , namely

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\*An alternative, consistent with the integration method used for  $L(x)$ , would be

$$\int_0^T e^{-r(x+t)} \ell(x+t) dt = \frac{T}{2} [e^{-rx} \ell(x) + e^{-r(x+T)} \ell(x+T)].$$

However, the difference introduced would be negligible.

$$\hat{T}_x^M = T_x^M \cosh\left(\frac{rT}{2}\right) + \frac{2}{T} \left[ \sinh\left(\frac{rT}{2}\right) - \frac{rT}{2} \right] \quad (7)$$

in which  $\cosh\left(\frac{rT}{2}\right)$  and  $\sinh\left(\frac{rT}{2}\right)$  are the hyperbolic cosine and hyperbolic sine of  $\frac{rT}{2}$ , respectively. (For a detailed derivation, see the Appendix).

Note that, in the case of the last age group,  $z$  years and over, the relationship in (7) should be replaced by (see Appendix):

$$\hat{T}_z^M = T_z^M e^{\frac{rT}{2}} - r. \quad (8)$$

From (7) and (8), it is clear that the stable death rates are only slightly different from the initial ones. The discrepancy between both rates is proportional to the intrinsic rate of growth  $r$ . Moreover, for all age intervals except the last, the stable death rate is always higher than its initial value by a quantity which depends on the square of  $r$ .

The preceding reasoning can be easily extended to the multi-regional case in which the counterpart of the death rate  $T_x^M$  is a matrix  $\tilde{T}_x^M$ , defined in Rogers and Ledent (1976), as:

$$\tilde{T}_x^M = \begin{bmatrix} 1_{T_x^M}^d + \sum_{\substack{j=1 \\ j \neq 1}}^n 1_{T_x^M}^j & - 2_{T_x^M}^1 & - n_{T_x^M}^1 \\ - 1_{T_x^M}^2 & 2_{T_x^M}^d + \sum_{\substack{j=1 \\ j \neq 2}}^n 2_{T_x^M}^j & \vdots \\ & \ddots & \vdots \\ - 1_{T_x^M}^n & \dots & n_{T_x^M}^d + \sum_{\substack{j=1 \\ j \neq n}}^n n_{T_x^M}^j \end{bmatrix} \quad (9)$$

where  $1_{T_x^M}^d$  is the death rate in age group  $(x, x + T)$  in region  $i$ .

$\frac{iM^j}{T\tilde{x}}$  is the migration rate for region  $i$  to region  $j$  in age group  $(x, x + T)$ .

The matrix  $T\tilde{x}^M$  of initial rates also may be defined in terms of life table statistics. The definition analogous to (4), is given in Rogers and Ledent (1976):

$$T\tilde{x}^M = [\tilde{\ell}(x) - \tilde{\ell}(x + T)] \tilde{L}(x)^{-1} \quad (10)$$

where  $\tilde{\ell}(x)$  and  $\tilde{L}(x)$  are the multiregional extensions of  $\ell(x)$  and  $L(x)$ . One may define a matrix of stable rates  $\hat{T\tilde{x}^M}$  such that:

$$\begin{aligned} \hat{T\tilde{x}^M} = [e^{-rx} \tilde{\ell}(x) - e^{-r(x+T)} \tilde{\ell}(x+T)] \\ [ \int_0^T e^{-r(x+t)} \tilde{\ell}(x+t) dt ]^{-1} - r\tilde{I} \end{aligned} \quad (11)$$

Generalizing the integration methods embodied in (5) and (6), and combining (10) and (11) finally yields:

$$\hat{T\tilde{x}^M} = T\tilde{x}^M \cosh\left(\frac{rT}{2}\right) + \frac{2}{T} \tilde{I} \left[ \sinh\left(\frac{rT}{2}\right) - \frac{rT}{2} \right] \quad (12)$$

and for the last age group\*,

$$\hat{T\tilde{z}^M} = T\tilde{z}^M e^{\frac{rT}{2}} - r\tilde{I} \quad (13)$$

The conclusion is that regional stable rates takes values only slightly different from their initial counterparts. The regional stable rates are related to the initial rates by formulas such as (7) and (8). If one assumes  $r$  to be small, this may be rewritten as follows (see the Appendix):

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\*Rogers' multiregion projection model (Rogers 1975) assumes zero outmigration rates for the last age group. However, this assumption is needlessly restrictive as shown by Schoen (1975). His definition of the last age group death and migration rates may be rewritten in a matrix form, by simply extending the definition of the last age group death rate of the single region case:

$$X\tilde{z}^M = \tilde{\ell}(z) \tilde{L}(z)^{-1}$$

$$\hat{i}_{T_x}^d \approx i_{T_x}^d + \frac{r^2 T^2}{8} (i_{T_x}^d + \frac{r}{3} + \frac{r^2 T^2}{48}) \quad (14)$$

for  $i = 1, \dots, n$  and  $x = 0, \dots, z - T$

and

$$\hat{i}_{T_z}^d \approx i_{T_z}^d - r [1 - \frac{T}{2} \frac{i_{T_z}^d}{T_z}] \quad \text{for } i = 1, \dots, n \quad (15)$$

In the case of outmigration rates, the relationship between initial and stable rates appear to be simpler:

$$\hat{i}_{T_x}^j \approx i_{T_x}^j (1 + \frac{r^2}{8} T^2) \quad (16)$$

for all  $i = 1, \dots, n$ ;  $j = 1, \dots, n$  ( $j \neq i$ ) and  $x = 0, \dots, z$

The conclusion here is that stable death rates (except for the last age group) and stable migration rates are higher than their initial values by a very small quantity that is proportional to the square of the intrinsic rate of growth  $r$ .

Table 1 presents a comparison of the initial and stable death and migration rates for selected age groups of the multiregional projection model based on 1961 data for Slovenia and the rest of Yugoslavia. There is clearly a higher discrepancy between initial and stable death rates in the last age group.

#### Intrinsic death (and migration rates)

In the single-region projection model, Keyfitz (1968) derives the intrinsic death rate  $d$  as the difference between the intrinsic birth rate  $b$  and the intrinsic growth rate  $r$ :

$$d = b - r \quad (17)$$

in which, by definition,

$$b = \sum_x T_x^F c(x) \quad (18)$$

where  ${}_T F_x$  is the age-specific fertility rate, assumed to be constant over the projection period, and  $c(x)$  is the fraction of the stable population in age group  $(x, x + T)$ .

Having previously derived the stable age-specific death rates, we now can obtain directly the intrinsic death rate  $d$  as

$$d = \sum_x \hat{T} M_x c(x) \quad (19)$$

In the multiregion projection model, regional intrinsic birth rates may be derived and computed from:

$$b_i = \sum_x \hat{T} F_x c_i(x) \quad (20)$$

where  $\hat{T} F_x$  is the region-specific fertility rate in age group  $x$  and  $c_i(x)$  is the fraction of the population of region  $i$  in that age group at stability.

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\*It is easy to check, from the previously derived formulas for  $\hat{T} M_x$ , that  $\sum_x \hat{T} M_x c(x)$  is equal to  $b - r$ . Noting that the denominator of the right-hand side term of  $\hat{T} M_x$  is equal to

$\frac{\ell(0)}{b} \cdot c(x)$ , we have that:

$$\hat{T} M_x c(x) = \frac{b}{\ell(0)} [e^{-rx} \ell(x) - e^{-r(x+T)} \ell(x+T)] - r c(x)$$

and,

$$\sum_x \hat{T} M_x c(x) = \frac{b}{\ell(0)} [\ell(0) - \cancel{e^{-rT} \ell(T)} + \cancel{e^{-rT} \ell(T)} - \cancel{e^{-2rT} \ell(2T)} + \cancel{e^{-2rT} \ell(2T)} \dots] - r \sum_x c(x)$$

The quantity between brackets reduces to  $\ell(0)$  so that

$$\sum_x \hat{T} M_x c(x) = b - r \sum_x c(x) = b - r.$$

Table 1

Initial and stable death and migration rates for selected age groups of the multiregional projection model based on 1961 data for Slovenia and the rest of Yugoslavia.\* (females only)

Age group	Region	Death Rate		Migration Rate	
		INITIAL	STABLE	INITIAL	STABLE
0	Slovenia	0.006150	0.006151	0.002832	0.002832
	Rest of Yugoslavia	0.022468	0.022471	0.000272	0.000272
20	Slovenia	0.000747	0.000747	0.007170	0.007171
	Rest of Yugoslavia	0.001220	0.001220	0.000937	0.000937
85+	Slovenia	0.203611	0.200640	0	0
	Rest of Yugoslavia	0.143486	0.139591	0	0

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\*Available data for this illustration indicate zero migration figures for the last age group, which explains why both observed and stable migration rates are equal to zero for the 85 and over group.



Unlike the single-region case, however, the residual from  $r$  only yields the net "absence" rate from region  $i$  and does not permit the identification of the regional intrinsic rates of death, out-migration and immigration. However, such intrinsic rates may be obtained directly from the knowledge of the age-specific mortality and migration rate  $\hat{M}_x^j$  and  $\hat{M}_x^j$ , respectively.

The intrinsic death and outmigration rates for region  $i$  can be defined and calculated using:

$$d_i = \sum_x \hat{M}_x^d c_i(x) \quad (21)$$

and

$$o_i = \sum_x \left[ \sum_{j \neq i} \hat{M}_x^j \right] c_i(x) \quad (22)$$

respectively. Of course, one could obtain the intrinsic immigration rate ( $i_i$ ) from the identity:

$$i_i = r - b_i + d_i - o_i$$

but a direct derivation is possible from the knowledge of the stable age-specific (out) migration rates and the stable regional populations.

The immigration flow into region  $i$  consists of the sum of all migration flows to region  $i$  from the other regions. If  $Y$  denotes the stable equivalent of the initial population of the system, and if  $y_j$  is the proportion of the stable population residing in  $j$ , then the migration flow into region  $i$ , at time  $t$ , say  $M_{.i}(t)$ , is equal to

$$Y \left[ \sum_{j \neq i} \left[ \sum_x \hat{M}_x^j c_j(x) y_j \right] e^{rt} \right]. \quad \text{Since } i_i = \frac{M_{.i}(t)}{Y y_i e^{rt}}$$

we have that:

$$i_i = \frac{\sum_{j \neq i} \left[ \sum_x \hat{M}_x^j c_j(x) y_j \right]}{y_i} \quad (23)$$

Table 2 displays the values of the intrinsic rate for each type of demographic event obtained by application of formulas (20) through (23) to the 1961 two-region female Yugoslavian population. A partial check of these values is possible by computing the sum  $b_i - d_i + i_i - o_i$  for each region and comparing them to the intrinsic growth rate, as permitted by the figures appearing in table 2.

Multiregional projection model: comparison of the stable age-specific migration rates with Stone's stable age-specific proportions

Previously, Stone (1968) proposed formulas for stable age-specific migration rates which differ from those indicated above. However, the rates presented by Stone were of a different nature. To call them rates was improper since they really had the same nature as the survivorship proportions in the multiregional growth operator: the migration rate from region  $i$  to region  $j$  relating to age group  $x$  is defined by Stone simply as the probability  $s_{ij}(x)$  of surviving in region  $j$ ,  $T$  years later, for those alive in region  $i$  in age group  $(x, x + T)$ . Rogers (1975) labeled Stone's quantities as  $T$ -year migration rates in contrast to age-specific migration rates such as  $\hat{M}_{ij}^x$ . He then used Stone's formulas as a starting point to derive  $T$ -year consolidated "crude"  $T$ -year stable migration rates (see Rogers 1975, p. 129 and 130) from which he drew some approximate values of the intrinsic migration rates. These appear in Table 3 by way of contrast.

In fact, there is no simple relationship linking the consolidated "crude"  $T$ -year stable rates of Stone and the corresponding intrinsic death and migration rates, which rules out the possibility of deriving the latter from the former without using any approximation.\*

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\*(see page 12 for footnote)

Table 2

Intrinsic Rates for the 1961 Two-Region  
Female Population System of Yugoslavia

	SLOVENIA	REST OF YUGOSLAVIA
$b_i$	0.015453	0.019009
$d_i$	0.011998	0.012753
$o_i$	0.002457	0.000301
$i_i$	0.005101	0.000145
$b_i - d_i + o_i - i_i$	0.006098	0.006100
$r$	0.006099	0.006099

Table 3

Approximate Intrinsic Rates derived by Rogers  
for the 1961 Two-Region Female Population System of Yugoslavia

	SLOVENIA	REST OF YUGOSLAVIA
$b_i$	0.01545	0.01901
$d_i$	0.01196	0.01275
$o_i$	0.00241	0.00030
$i_i$	0.00502	0.00014
$r$	0.00610	0.00610

Source: Rogers (1975) p. 131

## Conclusion

In both the single-region and multiregional models of population projection, the treatment of the death (migration) schedule contrasts with the treatment of the fertility schedule. On the one hand, mortality (migration) behavior results from the assumption of an unchanged curve of instantaneous transition forces; on the other hand, fertility behavior stems from an assumption of constant age-specific fertility rates, averaged over each unit projection period. Therefore, as a consequence of the change in age composition over the projection process, the instantaneous fertility curve is implicitly modified while the age-specific mortality and migration rates take on new values that, at stability, are only slightly different from their original values. Only the death rate for the last age group presents a relatively high discrepancy between its stable and initial values: this result stems from the straightforward application of the approximation contained in (6) to the last age group, which, instead of being  $T$  - years in length, is a half open interval.

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\*There exist, however, a relationship between the consolidated  $T$ -year crude birth rate and intrinsic rate of fertility. The birth rate a la Stone ( $T^b$ ) may be defined as

$$T^b = \frac{T}{2} \sum_x [{}_T F_x c(x) + s(x) {}_T F_{x+T}] c(x)$$
 in which  $s(x)$  is the survivorship proportion for the age group  $(x, x + T)$ .

$$T^b = \frac{T}{2} \left[ \sum_x {}_T F_x c(x) + \sum_x {}_T F_x + T [s(x) c(x)] \right]$$

Since  $s(x) c(x) = c(x + T) e^{rT}$ , we have:

$$T^b = \frac{T}{2} \left[ \sum_x {}_T F_x c(x) + e^{rT} \sum_x {}_T F_x + T c(x + T) \right] = \frac{T}{2} [b + e^{rT} b]$$

in which  $b$  is the intrinsic birth rate. The intrinsic birth rate thus may be obtained from the consolidated  $T$ -year "crude" birth rate by using:

$$b = \frac{T^b}{\frac{T}{2} [1 + e^{rT}]}$$

which in the case  $r$  is small, may be adequately approximated as

$$b \approx \frac{\sqrt{T}^b}{T(1 + \frac{rT}{2})}$$

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# APPENDIX

## Derivation of the Formula Expressing the Stable Age-Specific Death Rates in Terms of their Initial Values

Assuming that the denominator of the first term on the right-hand side of (3) may be approximated as indicated in (6), and dividing both numerator and denominator of the term by  $e^{-r(x + \frac{T}{2})}$  yields

$$\hat{T}M_x = \frac{e^{\frac{rT}{2}} \ell(x) - e^{-\frac{rT}{2}} \ell(x+T)}{L(x)} - r, \quad (24)$$

where  $\ell(y)$  is simply the number of survivors at exact age  $y$  in the life table population constructed from the initially chosen death rates. The following relationship links  $\ell(x)$  and  $\ell(x+T)$ :

$$\ell(x+T) = {}_T p(x) \ell(x) = [1 - {}_T q(x)] \ell(x) \quad (25)$$

in which  ${}_T q(x)$  is the probability of dying in the next  $T$  years for the survivors  $\ell(x)$  of the initial cohort at age  $x$ .

Using the formula relating  ${}_T q(x)$  to  $T M_x$  (Keyfitz 1968):

$${}_T q(x) = \frac{T T M_x}{1 + \frac{T}{2} T M_x}, \quad (26)$$

we have that

$$\ell(x+T) = \frac{1 - \frac{T}{2} T M_x}{1 + \frac{T}{2} T M_x} \ell(x) \quad (27)$$

Eliminating  $\ell(x+T)$  in (4) and (5) yields  $L(x)$  as a function of  $\ell(x)$  and  $T M_x$ :

$$L(x) = \frac{T}{1 + \frac{T}{2} T M_x} \ell(x) \quad (28)$$

Substituting the last two formulas in (3) and dividing both numerator and denominator of the first term on the right-hand side by  $\ell(x)$  yields:

$$T^{\hat{M}}_x = \frac{1 + \frac{T}{2} M(x)}{T} \left[ e^{\frac{rT}{2}} - e^{-\frac{rT}{2}} \frac{1 - \frac{T}{2} T^M_x}{1 + \frac{T}{2} T^M_x} \right] - r$$

which may be rearranged as:

$$T^{\hat{M}}_x = \frac{1}{T} \left[ \frac{T}{2} T^M_x \left( e^{\frac{rT}{2}} + e^{-\frac{rT}{2}} \right) + \left( e^{\frac{rT}{2}} - e^{-\frac{rT}{2}} \right) \right] - r$$

Observing that

$$\frac{e^{\frac{rT}{2}} + e^{-\frac{rT}{2}}}{2} = \cosh \left( \frac{rT}{2} \right),$$

and

$$\frac{e^{\frac{rT}{2}} - e^{-\frac{rT}{2}}}{2} = \sinh \left( \frac{rT}{2} \right), \text{ we have that}$$

$$T^{\hat{M}}_x = T^M_x \cosh \left( \frac{rT}{2} \right) + \frac{2}{T} \left[ \sinh \left( \frac{rT}{2} \right) - \frac{rT}{2} \right] \quad (7)$$

Since  $r$  is generally small, the following approximation holds:

$$\cosh \left( \frac{rT}{2} \right) \approx 1 + \frac{r^2}{8} T^2 + \frac{r^4}{384} T^4$$

Hence,

$$T^{\hat{M}}_x \approx T^M_x \left( 1 + \frac{r^2}{8} T^2 + \frac{r}{384} T^4 \right) + \frac{r^3}{24} T^2$$

so that the difference between the stable and initial death rates is equal to:

$$T^{\hat{M}}_x - T^M_x \approx \frac{r^2 T^2}{8} \left( T^M_x + \frac{r}{3} + \frac{r^2 T^2}{48} \right) \quad (29)$$

This formula indicated that the discrepancy between the two rates is positive and small, and it is proportional to the square of the intrinsic rate  $r$ .

In the case of the last age group, a relationship linking  ${}_T\hat{M}_Z$  to  ${}_T^M_Z$  may be obtained directly by combining the definitional equations for these rates and using the approximation (6):

$${}_T\hat{M}_Z = {}_T^M_Z e^{\frac{rT}{2}} - r, \quad (8)$$

which for  $r$  small, leads to the approximate equality,

$${}_T\hat{M}_Z \approx {}_T^M_Z - r \left[ 1 - \frac{T}{2} {}_T^M_Z \right] \quad (30)$$

Again, the stable death rate does not differ much from the initial one. However, unlike in the general case, the stable death rate is generally smaller than the observed death rate (i.e.,  ${}_T\hat{M}_Z$  is less than  ${}_T^M_Z$  if  ${}_T^M_Z < \frac{2}{T}$  ).

Moreover, the discrepancy between both rates is proportional to the intrinsic rate  $r$  rather than to its square.



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July 1977

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